energy input distributions The accelerator lengths were shorter than those obtained from several other physically realistic analytic solutions but obviously did not represent a mathematical minimum It is therefore of interest to compare the present solution with that in Ref 2

First, it can be seen that, for $\epsilon \ll 1$ and $(\epsilon x)^2 \ll 1$, both solutions are identical provided $n = \frac{1}{2}$ in Ref 2 [Compare Eq (23) with Eqs (15b) and (18a)] For larger ϵ and ϵx, the comparison is more complex and requires numerical computation In particular, for large ϵ, one should integrate Eq (8) numerically rather than use the analytic perturbation solution developed herein Also, if the minimum-length solution is to be compared with that in Ref 2 for the same inlet and exit conditions (in terms of p_1, u, ρ, $\omega\tau$), it is necessary to determine the value of n in Ref 2 such that $B_2 = 1$ Several examples have indicated that the accelerator lengths agree within 10% when n is determined as described in the foregoing However, the distribution of jE_yA, A, and B was different for the two solutions A more detailed comparison is beyond the scope of the present note

The minimum-length accelerator solution developed herein, and in Ref 1, is based upon the assumptions of a perfect gas and constant enthalpy, B and σ If these assumptions were relaxed, particularly those regarding the constant enthalpy and B, different minimum lengths would result Physical considerations, such as Hall effects, ion slip, and the need for relatively uniform area variations, may provide other guide lines for minimizing Eq (7) Note that the area variation in the present solution [Eq (17a)] may not permit shock-free supersonic flow Hence, the present solution and that of Ref 1 provide only a first estimate for the design of a physically realistic "minimum-length" accelerator with relatively uniform enthalpy and B It may be necessary to modify this design and then to determine the performance by integrating the full one-dimensional equations of motion

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Instability of Three-Row Vortex Streets

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CLOUD patterns resembling vortices on the leeward side of islands have been observed[1] It is well known that two rows of vortex streets, known as Kármán vortex streets, exist behind a single bluff body representing a highly elevated island in a uniform stream On the leeward side of a group island, it might be expected that more than two rows of vortex streets may exist It is of interest, therefore, to study the possible stable configuration of three rows of vortex streets and their stability criteria The three rows of vortex streets may conceivably appear on the leeward side of two islands with small openings between them It will be shown in this note that there exists a steady configuration of three-row vortex streets, and it is always unstable because a special small disturbance can be found which makes the configuration unstable

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Steady-State Solution

Consider three parallel infinite rows of the same spacing, say a, with a total strength equal to zero, so arranged that the origin of the coordinates is chosen at any midpoint of two vortices in the second row, and that the x axis coincides with that row The coordinates of each vortex in the first, second, and third rows will be $(ma + b, h_1)$, $[na + (a/2), 0]$ and $(pa + c, -h_2)$, respectively, where m, n, and $p = 0, \pm 1, \pm 2,$ The strengths of the three rows of vortices are Γ_1, $-(\Gamma_1 + \Gamma_2)$, Γ_2, respectively The sign of Γ_1 and Γ_2 are arbitrary, but the distances h_1 and h_2 are positive with the middle row chosen as X axis The velocity of vortex[2] at point $z_1 = b + ih_1$, $z_2 = a/2$, and $z_3 = c - ih_2$ may be written as

$$v_1 = -i(\Gamma_1 + \Gamma_2)\frac{\pi}{a}\tan\frac{\pi}{a}(b + h_1 i) - i\Gamma_2\frac{\pi}{a}\cot\frac{\pi}{a}[b - c + (h_1 + h_2)i]$$

$$v_2 = i\Gamma_1\frac{\pi}{a}\tan\frac{\pi}{a}(b + h_1 i) - i\Gamma_2\frac{\pi}{a}\tan\frac{\pi}{a}(c - ih_2)$$

$$v_3 = i\Gamma_1\frac{\pi}{a}\cot\frac{\pi}{a}[c - b - (h_1 + h_2)i] + i(\Gamma_1 + \Gamma_2)\frac{\pi}{a}\tan\frac{\pi}{a}(-c + h_2 i)$$

The steady-state solution requires that $v_1 = v_2 = v_3$, i e ,

$$\tan\frac{\pi}{a}(b + h_1 i) + \cot\frac{\pi}{a}[b - c + (h_1 + h_2)i] = \tan\frac{\pi}{a}(c - h_2 i) \quad (1)$$

$$\tan\frac{\pi}{a}(b + h_1 i) = \cot\frac{\pi}{a}[c - b - (h_1 + h_2)i] - \tan\frac{\pi}{a}(-c + h_2 i) \quad (2)$$

Note that Eqs (1) and (2) are the same To find the values of b and c, we separate Eq (1) into real and imaginary parts[3] Thus they are the following cases

***Case 1:* $b = 0, c = 0$**

Equation (1) reduces to

$$\tanh\frac{h_1\pi}{a} - \coth\frac{(h_1 + h_2)\pi}{a} = -\tanh\frac{h_2\pi}{a} \quad (3)$$

Let $\tanh(h_1\pi/a) = A$ and $\tanh(h_2\pi/a) = B$; then $0 < A < 1$, $0 < B < 1$ for $h_1 > 0$, $h_2 > 0$ Equation (3) gives

$$A^2 + B^2 = 1 - AB \quad (4a)$$

or

$$B = [-A \pm (4 - 3A^2)^{1/2}]/2 \quad (4b)$$

Since $0 < A < 1$, $A < 1 < (4 - 3A^2)^{1/2}$, then the only root of Eq (4b) such that $0 < B < 1$ will be $[-A \pm (4 - 3A^2)^{1/2}]/2$

***Case 2:* $b = a/2$, $c = 0$**

Equation (1) reduces to

$$-(1/A) + [(A + B)/(AB + 1)] = B \quad (5)$$

Therefore, $B = -1 \pm (4A^2 - 3)^{1/2}/2A$ The roots of B in Eq (5) are both negative because $(4A^2 - 3)^{1/2} < 1$ Hence, no steady solution exists for $b = a/2$, $c = 0$, since we pick h_2 as positive

***Case 3:* $b = 0$, $c = a/2$**

Equation (1) reduces to $-A + [(A + B)/(1 + AB)] = 1/B$ This is the same as Eq (5) if we interchange A and B; hence, a steady-state solution does not exist

Case 4: $b = a/2$, $c = a/2$

Equation (1) reduces to

$$(A + B)^2 = AB(1 + AB) \tag{6}$$

or

$$A^2 + B^2 = AB(AB - 1) \tag{7}$$

Consider the ratio of Eqs (7) and (8) The left-hand side is positive and the right-hand side is $[(AB + 1)/(AB - 1)] < 0$ for $0 < A < 1$, $0 < B < 1$, which leads to a contradiction Therefore, no steady solution exists when $b = a/2$, $c = a/2$

From the preceding discussion, only case 1 has one steady solution, which is the standard configuration with $b = c = 0$; that is, the vortices of the first row and the third row are located directly above and below the midpoint of the line joining two vortices of the middle row The vortices in all three rows are moving with the same velocity along the x direction, i e, $v_1 = v_2 = v_3 = -\pi/a[\Gamma_1 A - \Gamma_2 B]$, and the distances h_1 and h_2 between the rows are related by Eq (4)

Stability of Steady-State Solution

We examine the stability of this steady-state solution by observing its variation at time $t > 0$ If we displace each vortex slightly, those in the first, second, and third rows will move to $ma + vt + h_1 i + z_m$, $na + (a/2) + vt + z_n'$, and $pa + vt - h_2 i + z_p''$ respectively, where $|z_m|$, $|z_n'|$, and $|z_p''|$ are all small in quantity The system will be stable if the quantities will remain small We put $z_m = \gamma \cos m\theta$, $z_n' = \gamma' \cos(n + \frac{1}{2})\theta$, $z_p'' = \gamma'' \cos p\theta$, where γ, γ', and γ'' are small complex numbers Define $K_1 = h_1/a$, $K_2 = h_2/a$, $K_3 = (h_1 + h_2)/a$, and $\Gamma = \Gamma_1/\Gamma_2$ Here we assume $\Gamma_2 \neq 0$ The special case $\Gamma_2 = 0$ will be discussed at the end of this section Then, the equations governing the motion γ, γ', and γ'' are

$$\frac{d^2\gamma}{dt^2} = \frac{4\Gamma_2^2}{a^4}\{[A_{10} - (1 + \Gamma)A_{21} + A_{13}]^2 - \Gamma(1 + \Gamma)C_{21}^2 + \Gamma C_{13}^2\}\gamma + \frac{4\Gamma_2^2}{a^4}\{(1 + \Gamma)C_{21}[-A_{10} - A_{21} + A_{22} + A_{13}] - (1 + \Gamma)C_{13}C_{22}\}\gamma' + \frac{4\Gamma_2^2}{a^4}\{-(1 + \Gamma)C_{13}[A_{10} - A_{21} - A_{22} + A_{13}] - (1 + \Gamma)C_{21}C_{22}\}\gamma''$$

and similarly for γ' and γ'', where

$$A_{10} = \sum_{m=1}^{\infty} \frac{1 - \cos m\theta}{m^2} = \frac{1}{4}\theta(\pi - \theta)$$

$$A_{21} = \sum_{m=0}^{\infty} \frac{(m + \frac{1}{2})^2 - K_1^2}{[(m + \frac{1}{2})^2 + K_1^2]^2} = \frac{\pi^2}{2\cosh^2 K_1\pi}$$

$$A_{22} = \sum_{m=0}^{\infty} \frac{(m + \frac{1}{2})^2 - K_2^2}{[(m + \frac{1}{2})^2 + K_2^2]^2} = \frac{\pi^2}{2\cosh^2 K_2\pi}$$

$$A_{13} = \sum_{m=0}^{\infty} \frac{m^2 - K_3^2}{[m^2 + K_3^2]^2} = \frac{-\pi^2}{2\sinh^2 K_3\pi}$$

$$C_{21} = \sum_{m=0}^{\infty} \frac{(m + \frac{1}{2})^2 K_1^2}{[(m + \frac{1}{2})^2 + K_1^2]^2}\cos(m + \tfrac{1}{2})\theta$$

$$C_{22} = \sum_{m=0}^{\infty} \frac{(m + \frac{1}{2})^2 - K_2^2}{[(m + \frac{1}{2})^2 + K_2^2]^2}\cos(m + \tfrac{1}{2})\theta$$

$$C_{13} = \sum_{m=0}^{\infty} \frac{m^2 - K_3^2}{[m^2 + K_3^2]^2}\cos m\theta$$

We then substitute $\gamma = \exp 2\Gamma_2\lambda t/a^2$, $\gamma' = \exp 2\Gamma_2\lambda t/a^2$, and $\gamma'' = \exp 2\Gamma_2\lambda t/a^2$ The characteristic equation will be

$$\lambda^6 + F\lambda^4 + G\lambda^2 + H = 0 \tag{8}$$

where F, G, and H are given in Ref 3

For the configuration to be stable, it should be stable with respect to all modes of small disturbances, i e, all of the characteristic roots λ of Eq (8) should not have positive real parts for any value of θ On the other hand, if there exists a mode of small disturbance, e g, a special value of θ such that Eq (8) has one or more roots which are not real and negative regardless of the combinations of Γ_1/Γ_2 and h_1/h_2, then the configuration is definitely unstable Let us consider the special mode $\theta = \pi$ With $A_{10} = \pi^2/4$, $C_{12} = C_{22} = 0$, Eq (8) reduces to

$$(\lambda^2 - m^2)[\lambda^4 - (n^2 + l^2 + 2\Gamma C_{13}^2)\lambda^2 + (l^2 + \Gamma C_{13}^2)(n^2 + \Gamma C_{13}^2) - \Gamma C_{13}^2(l + n)^2] = 0 \tag{9}$$

where

$$m = -(1 + \Gamma)A_{10} + \Gamma A_{21} + A_{22}$$

$$l = \Gamma A_{10} - (1 + \Gamma)A_{21} + A_{13}$$

$$n = A_{10} + \Gamma A_{13} - (1 + \Gamma)A_{22}$$

Hence,

$$\lambda^2 - m^2 = 0 \tag{10a}$$

or

$$\lambda^4 - (n^2 + l^2 + 2\Gamma C_{13}^2)\lambda^2 + (l^2 + \Gamma C_{13}^2)(n^2 + \Gamma C_{13}^2) - \Gamma C_{13}^2(n + l)^2 = 0 \tag{10b}$$

From Eq (10a), $\lambda^2 = m^2$ if λ^2 has to be real and negative $m = -(1 + \Gamma)A_{10} + \Gamma A_{21} + A_{22}$ must vanish; hence,

$$\Gamma = \frac{2\,\mathrm{sech}^2 K_2\pi - 1}{1 - 2\,\mathrm{sech}^2 K_1\pi} = \frac{1 - 2B^2}{2A^2 - 1} \tag{11}$$

The coefficient of λ^2 in Eq (10b), $-[n^2 + l^2 + 2\Gamma C_{13}^2]$, is evidently negative if $\Gamma_1/\Gamma_2 = \Gamma > 0$ Hence, at least one of the characteristic roots of Eq (10b) or Eq (9) will not be real and negative We conclude that, for any combination of Γ_1 and Γ_2, provided they are of the same sign (i e, $\Gamma_1/\Gamma_2 = \Gamma > 0$), the steady configuration is unstable

When Γ_1 and Γ_2 are of the opposite sign (i e, $\Gamma = \Gamma_1/\Gamma_2 < 0$), we show in the following that the coefficient of λ^2 in Eq (10b), $-[n^2 + l^2 - 2|\Gamma|C_{13}^2]$, remains negative Since $m = 0$, $n^2 = (n + m)^2 = \Gamma^2(A_{10} - A_{21} + A_{22} - A_{13})^2$, $l^2 = (l + m)^2 = (A_{10} + A_{21} - A_{22} - A_{13})^2$, and $A_{10} = \pi^2/4$, $A_{21} = (\pi^2/2)(1 - A^2)$, $A_{22} = (\pi^2/2)(1 - B^2)$, $A_{13} = -(\pi^2/2)AB$, and $C_{13} = -(\pi^2/2)(AB)^{1/2}$ It follows that

$$n^2 + l^2 - 2|\Gamma|C_{13}^2 = (\pi^4/4)\{[(\tfrac{1}{2} + AB)^2(1 - |\Gamma|) - (A^2 - B^2)(1 + |\Gamma|)]^2 + 2|\Gamma|[\tfrac{1}{4} + A^2B^2 - (A^2 - B^2)]^2\} \tag{12}$$

From Eq (11) it is evident that either $A^2 < \frac{1}{2}$, $B^2 < \frac{1}{2}$ or $A^2 > \frac{1}{2}$, $B^2 > \frac{1}{2}$ in order that Γ be negative If $A^2 > \frac{1}{2}$, $B^2 > \frac{1}{2}$, the left-hand side of Eq (4a), $(A^2 + B^2)$, is larger than one, whereas the right-hand side, $(1 - AB)$, is smaller than one, which leads to a contradiction Hence, it is necessary that $A^2 < \frac{1}{2}$, $B^2 < \frac{1}{2}$ for $\Gamma < 0$ With $\frac{1}{4} > (A^2 - B^2)^2$, it follows from Eq (12) that $-[n^2 + l^2 - 2|\Gamma|C_{13}^2] < 0$ It is therefore concluded that the three parallel rows of vortices are always unstable for any finite value of Γ

For the special case $\Gamma_2 = 0$, i e, $\Gamma \to \infty$, the third-row vortex disappears, and one can drop the stability of motion requirement of the third row That is, γ'' is omitted in the perturbation equations for γ and γ' The stability condition reduces to the Kármán criterion

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